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New integrable extension of the Hubbard chain with variable range hopping

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Abstract. A new integrable variant of the one-dimensional Hubbard model with variable-range correlated hopping is studied. The Hamiltonian is constructed by applying the quantum inverse scattering method on the infinite interval at zero density to the one-parameter deformation of the L-matrix of the Hubbard model. By construction, this model has $\mathrm{Y(su(2))} \oplus \mathrm{Y(su(2))}$ symmetry in the infinite chain limit. Multiparticle eigenstates of the model are investigated by using this method.

1. Introduction

As a model of strongly correlated electrons, the Hubbard model has been attracting much interest in solid state physics. In particular, in one dimension, the model is exactly solvable [1] and its thermodynamic properties can be calculated out, which give a good testing ground for theories of strongly correlated electron systems. From the viewpoint of integrability and algebraic aspects of the one-dimensional Hubbard model, there have been many works, including the pioneering work of the coordinate Bethe ansatz by Lieb and Wu [2], the quantum inverse scattering method [3–6], its SO(4) invariance [7–9], $\mathrm{Y(su(2))} \oplus \mathrm{Y(su(2))}$ invariance in the infinite chain limit [10] and the recent development of the algebraic and analytic Bethe ansatz [11, 12]. Recently, there has been increasing interest in such algebraic aspects of the integrable systems, especially in quantum groups. For example, Hikami [13] and Bouwknegt and Schoutens [14] studied Yangian symmetry underlying in the Haldane–Shastry spin chain and obtained its character formula. Karowski and Zapletal [15] invented an $U_q(\mathrm{sl}(n))$-invariant model, and Gould *et al* [16] studied quantum superalgebra to obtain new integrable models of correlated electrons.

Let us return to the one-dimensional Hubbard model. One of the novel properties of the R-matrix $R(\lambda, \mu)$ of the model is that it is thought to be impossible to express it as a function of a difference of two spectral parameters λ and μ. This lack of the 'difference property' has prevented us from investigating underlying integrable structures of the model. For example, it is not known whether this R-matrix is expressible as an intertwiner of a certain algebra. Since the methods for calculating various correlation functions known so far [17, 18] requires an understanding of such underlying structures of the model to some extent, it is necessary to deepen our knowledge of the mathematical structures of the Hubbard model in order to calculate correlation functions.

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Although, at first glance, the absence of the ‘difference property’ merely complicates the situation, it gives rise to a one-parameter integrable extention of the L-matrix and the Hamiltonian of the Hubbard model as was noticed by Shiroishi and Wadati [19]. This extended Hubbard model is interpreted as an electronic model with on-site and neighbouring-site interactions and correlated hopping to the neighbouring sites. Although the form of the Hamiltonian is complicated and thus is difficult to be equipped with physical meaning, it can be of some help with the understanding of the structures of the original Hubbard model.

On the other hand, we have recently discovered that the R-and L-matrices of the one-dimensional Hubbard model can be put into a formulation of quantum inverse scattering method (QISM) on an infinite interval [20, 21], which has been applied to other integrable models [22, 23]. Using that method, we can derive the existence of Yangian symmetry $\mathrm{Y(su(2))} \oplus \mathrm{Y(su(2))}$ and construct n-particle states upon zero-density vacuum. Based on this work, the aim of this paper is to put the one-parameter deformed L-matrix, which is described in the previous paragraph, into the same formulation of the QISM on an infinite interval. Through this procedure, a new electronic Hamiltonian with variable-range correlated hopping arises. It can be embedded in a family of an infinite number of commuting operators and thus is interpreted as a one-parameter integrable deformation of the Hubbard chain. As is the case for the usual Hubbard chain, Yangian invariance of the Hamiltonian and construction of multiparticle states can be directly established as a by-product of this method.

This paper is organized as follows. In section 2 we shall explain the integrability of the Hubbard chain of finite length and a one-parameter deformation of that model. Section 3 is devoted to the application of the QISM on an infinite interval to the one-parameter deformation of the L-matrix. Its resulting new Hamiltonian and commuting conserved operators are developed in section 4. The Yangian $\mathrm{Y(su(2))} \oplus \mathrm{Y(su(2))}$ invariance of the model follows by construction. In section 5, we shall construct multiparticle states upon the zero-density vacuum by use of symmetries and algebras of some operators. Section 6 contains concluding remarks and discussions.

2. Hamiltonian and monodromy matrix on the finite interval

The Hamiltonian for the one-dimensional Hubbard model is

$$\hat{H} = -\sum_{j,\sigma=\uparrow,\downarrow}(c_{j+1,\sigma}^{\dagger}c_{j,\sigma} + c_{j,\sigma}^{\dagger}c_{j+1,\sigma}) + U\sum_{j}(n_{j\uparrow} - \tfrac{1}{2})(n_{j\downarrow} - \tfrac{1}{2}) \qquad (2.1)$$

where $c_{j\sigma}$ and $c_{j\sigma}^{\dagger}$ are respectively the fermion annihilation and creation operators which satisfy the usual anticommutation relations, and $n_{j\sigma} = c_{j\sigma}^{\dagger}c_{j\sigma}$ is the particle number operator. This Hamiltonian has a large symmetry. First, it is invariant under the partial particle–hole transformation:

$$c_{j\uparrow} \to c_{j\uparrow} \qquad c_{j\downarrow} \to (-1)^{j}c_{j\downarrow}^{\dagger} \qquad U \to -U. \qquad (2.2)$$

Second, it is invariant under SO(4) algebra generated by S^{a} and η^{a} ($a = x, y, z$). S^{a} is the usual spin-SU(2) operator defined by $S^{a} = \frac{1}{2}\sum_{j}\sigma_{\alpha\beta}^{a}c_{j\alpha}^{\dagger}c_{j\beta}$, and η^{a} is obtained by performing the partial particle–hole transformation (2.2) to S^{a}. If we go to the infinite chain limit, the symmetry is enhanced to $\mathrm{Y(su(2))} \oplus \mathrm{Y(su(2))}$ Yangian [10].

The integrability of the fermionic Hubbard model (2.1) is based on a local exchange relation [6]

$$\check{\mathcal{R}}(\lambda,\mu)[\mathcal{L}_{j}(\lambda) \otimes_{\mathrm{s}} \mathcal{L}_{j}(\mu)] = [\mathcal{L}_{j}(\mu) \otimes_{\mathrm{s}} \mathcal{L}_{j}(\lambda)]\check{\mathcal{R}}(\lambda,\mu) \qquad (2.3)$$

where $\otimes_{\mathrm{s}}$ denotes the Grassman tensor product

$$[A \otimes_{\mathrm{s}} B]_{\alpha\gamma,\beta\delta} = (-1)^{[P(\alpha)+P(\beta)]P(\gamma)} A_{\alpha\beta} B_{\gamma\delta} \tag{2.4}$$

with the grading $P(1) = P(4) = 0$, $P(2) = P(3) = 1$. The expressions for the matrices $\check{\mathcal{R}}$ and $\mathcal{L}$ are presented in the appendix†. In these expressions, we use a function $h(\lambda)$ defined by

$$\frac{\sinh 2h(\lambda)}{\sin 2\lambda} = \frac{U}{4}. \tag{2.5}$$

The Hamiltonian (2.1) is reproduced by a logarithmic derivative of the transfer matrix $\tau_{mn}(\lambda)$;

$$\hat{H} = \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \tau_{mn}(\lambda)|_{\lambda=0} \qquad \tau_{mn}(\lambda) = \mathrm{str}(\mathcal{T}_{mn}(\lambda)) \equiv \mathrm{tr}((\sigma^z \otimes \sigma^z)\mathcal{T}_{mn}(\lambda)) \tag{2.6}$$

where the monodromy matrix $\mathcal{T}$ is given by

$$\mathcal{T}_{mn}(\lambda) = \mathcal{L}_{m-1}(\lambda)\mathcal{L}_{m-2}(\lambda)\ldots\mathcal{L}_n(\lambda)(m > n) \tag{2.7}$$

and str in (2.6) denotes the supertrace or the graded trace.

There is known to be an integrable spin chain equivalent to the Hubbard model [3, 4]. If we apply the Jordan–Wigner transformation

$$c_{j\uparrow} = \sigma_n^z \ldots \sigma_{j-1}^z \sigma_j^- \qquad c_{j\downarrow} = (\sigma_n^z \ldots \sigma_{m-2}^z \sigma_{m-1}^z)\tau_n^z \ldots \tau_{j-1}^z \tau_j^- \tag{2.8}$$

where σ and τ are the Pauli matrices, and $\sigma_j^{\pm} = (\sigma_j^x \pm \mathrm{i}\sigma_j^y)/2$, $\tau_j^{\pm} = (\tau_j^x \pm \mathrm{i}\tau_j^y)/2$, we obtain an equivalent spin model

$$\hat{H} = \sum_j (\sigma_{j+1}^+ \sigma_j^- + \sigma_j^+ \sigma_{j+1}^-) + \sum_j (\tau_{j+1}^+ \tau_j^- + \tau_j^+ \tau_{j+1}^-) + \frac{U}{4} \sum_j \sigma_j^z \tau_j^z. \tag{2.9}$$

Its integrability is supported by the spin-chain counterpart of the exchange relation [4]

$$\check{R}(\lambda, \mu)[L_j(\lambda) \otimes L_j(\mu)] = [L_j(\mu) \otimes L_j(\lambda)]\check{R}(\lambda, \mu). \tag{2.10}$$

Here $\check{R}(\lambda, \mu)$ and $L_j(\lambda)$, whose expressions can be found in [6], are the spin-chain counterparts of $\check{\mathcal{R}}(\lambda, \mu)$ and $\mathcal{L}_j(\lambda)$, respectively. As Olmedilla *et al* [6] found, the exchange relation of the fermionic model, (2.3), and that of the spin chain, (2.10), can be transformed into each other.

One of the peculiarities on the integrability of the Hubbard model that has been known for years is that the $\check{\mathcal{R}}$-matrix, or equivalently the $\check{R}$-matrix, is believed to lack the difference property, i.e. it is not a function of $\lambda - \mu$, nor can it be expressed as $f(\lambda) - f(\mu)$ with some function f. This has been an obstruction for further investigations of the underlying mathematical structures of the Hubbard model. However, on the other hand, the lack of the difference property allows us to consider a one-parameter integrable deformation of the Hubbard model, as noted by Shiroishi and Wadati [19]. This works as follows. By using the Yang–Baxter equation

$$R_{12}(\lambda, \mu)R_{13}(\lambda, \nu)R_{23}(\mu, \nu) = R_{23}(\mu, \nu)R_{13}(\lambda, \nu)R_{12}(\lambda, \mu) \tag{2.11}$$

where $R_{ij}(\lambda, \mu) = P_{ij}\check{R}_{ij}(\lambda, \mu)$ and P_{ij} is the transposition ($P(x \otimes y) = y \otimes x$), a new L-matrix defined by

$$L_1(\lambda)_\nu = R_{13}(\lambda, \nu) \tag{2.12}$$

† The matrix $\mathcal{R}$ in [6, 20, 21] is written as $\check{\mathcal{R}}$ in this paper, following the standard notation. It should not be confused with the matrix written in the standard notation as $\mathcal{R} = \mathcal{P}\check{\mathcal{R}}$.

satisfies an exchange relation

$$R_{12}(\lambda,\mu)L_1(\lambda)_\nu L_2(\mu)_\nu = L_2(\mu)_\nu L_1(\lambda)_\nu R_{12}(\lambda,\mu). \tag{2.13}$$

It can alternatively be written as

$$\check{R}(\lambda,\mu)[L(\lambda)_\nu \otimes L(\mu)_\nu] = [L(\mu)_\nu \otimes L(\lambda)_\nu]\check{R}(\lambda,\mu). \tag{2.14}$$

Considering that $L(\lambda)_{\nu=0} \propto L(\lambda)$, which can be checked by a direct calculation, we can say that $L(\lambda)_\nu$ is a one-parameter deformation of the L-matrix $L(\lambda)$ of the original Hubbard model. This new L-matrix can be used to produce a new Hamiltonian [19]. By using the monodromy matrix given by

$$T_{mn}(\lambda)_\nu = L_{m-1}(\lambda)_\nu L_{m-2}(\lambda)_\nu \ldots L_n(\lambda)_\nu \qquad (m > n) \tag{2.15}$$

the new Hamiltonian is given by

$$\begin{aligned}\hat{H}_\nu = \frac{\mathrm{d}}{\mathrm{d}\lambda}\ln\mathrm{tr}(T_{mn}(\lambda)_\nu)|_{\lambda=\nu} = -\sum_{j,\sigma=\uparrow,\downarrow}(c^\dagger_{j+1,\sigma}c_{j,\sigma} + c^\dagger_{j,\sigma}c_{j+1,\sigma})\\ +\frac{U}{4\cosh 2h(\nu)}\sum_j(2n_{j\uparrow}\cos^2\nu - 2n_{j+1\uparrow}\sin^2\nu\\ +\sin 2\nu(c^\dagger_{j\uparrow}c_{j+1\uparrow} - c^\dagger_{j+1\uparrow}c_{j\uparrow}) - \cos 2\nu)\\ \times(2n_{j\downarrow}\cos^2\nu - 2n_{j+1\downarrow}\sin^2\nu + \sin 2\nu(c^\dagger_{j\downarrow}c_{j+1\downarrow} - c^\dagger_{j+1\downarrow}c_{j\downarrow}) - \cos 2\nu).\end{aligned} \tag{2.16}$$

We have performed the Jordan–Wigner transformation to get a fermionic Hamiltonian. The reason for choosing the special value $\lambda = \nu$ is to obtain a local Hamiltonian. This Hamiltonian is Hermitian if ν is pure imaginary, and in that case it represents a model with on-site and neighbouring-site interaction and correlated hopping to the neighbouring sites. The exchange relation (2.14) results in

$$[\ln\mathrm{tr}\,T_{mn}(\lambda)_\nu, \ln\mathrm{tr}\,T_{mn}(\mu)_\nu] = 0. \tag{2.17}$$

Therefore, the Hamiltonian is embedded in a family of infinite number of commuting operators, and so the model (2.16) is integrable. The model (2.16) includes the Hubbard model (2.1) as in the $\nu = 0$ case.

So far we have discussed a spin chain version of the exchange relation (2.14). We can employ the same procedure used in [6] to fermionize the exchange relation of the spin chain (2.14), and we obtain

$$\check{\mathcal{R}}(\lambda,\mu)[\mathcal{L}_j(\lambda)_\nu \otimes_{\mathrm{s}} \mathcal{L}_j(\mu)_\nu] = [\mathcal{L}_j(\mu)_\nu \otimes_{\mathrm{s}} \mathcal{L}_j(\lambda)_\nu]\check{\mathcal{R}}(\lambda,\mu). \tag{2.18}$$

The explicit form of $\mathcal{L}_j(\lambda)_\nu$ is presented in the appendix. The exchange relation (2.14) and (2.18) can be considered as a one-parameter deformation of (2.10) and (2.3), respectively. Note that $\check{R}(\lambda,\mu)$ or $\check{\mathcal{R}}(\lambda,\mu)$ is unchanged by this one-parameter deformation.

3. Passage to the infinite interval

Next we pass to the infinite interval limit using the new exchange relation (2.18). The method is identical with the one in our previous works on the original Hubbard model [20, 21]. Let

$$\mathcal{T}_{mn}(\lambda)_\nu = \mathcal{L}_{m-1}(\lambda)_\nu\mathcal{L}_{m-2}(\lambda)_\nu \ldots \mathcal{L}_n(\lambda)_\nu \tag{3.1}$$

$$\mathcal{T}^{(2)}_{mn}(\lambda,\mu)_\nu = \mathcal{T}_{mn}(\lambda)_\nu \otimes_{\mathrm{s}} \mathcal{T}_{mn}(\mu)_\nu \tag{3.2}$$

where $\mathcal{T}_{mn}(\lambda)_\nu$ is a monodromy matrix on the finite interval. To consider the infinite-chain limit of the monodromy matrix, we should split off the asymptotics of its vacuum expectation value for $m, -n \to \infty$. Hence, this procedure is restricted to uncorrelated vacua, with which one can calculate vacuum expectation value of the monodromy matrix. Among four uncorrelated vacua, in which the electron density of each spin is either zero or unity, we take the zero-density vacuum $|0\rangle$ to renormalize the monodromy matrix. We use the following two matrices

$$V(\lambda)_\nu = \langle 0|\mathcal{L}_j(\lambda)_\nu|0\rangle \qquad V^{(2)}(\lambda,\mu)_\nu = \langle 0|\mathcal{L}_j(\lambda)_\nu \otimes_{\mathrm{s}} \mathcal{L}_j(\mu)_\nu|0\rangle \tag{3.3}$$

in order to normalize $\mathcal{T}_{mn}(\lambda)$ and $\mathcal{T}^{(2)}_{mn}(\lambda,\mu)$, respectively;

$$\tilde{\mathcal{T}}(\lambda)_\nu = \lim_{m,-n\to\infty} V(\lambda)_\nu^{-m}\mathcal{T}_{mn}(\lambda)_\nu V(\lambda)_\nu^n \tag{3.4}$$

$$\tilde{\mathcal{T}}^{(2)}(\lambda,\mu)_\nu = \lim_{m,-n\to\infty} V^{(2)}(\lambda,\mu)_\nu^{-m}\mathcal{T}^{(2)}_{mn}(\lambda,\mu)_\nu V^{(2)}(\lambda,\mu)_\nu^{n}. \tag{3.5}$$

These limits converge in the weak sense, though the matrices $\mathcal{T}_{mn}(\lambda)_\nu$ and $\mathcal{T}^{(2)}_{mn}(\lambda,\mu)_\nu$ do not have a definite limit when $m, -n \to \infty$. We shall call $\tilde{\mathcal{T}}(\lambda)_\nu$ a monodromy matrix on the infinite interval. It allows an alternative definition:

$$\tilde{\mathcal{T}}(\lambda)_\nu = I_4 + \sum_j (\tilde{\mathcal{L}}_j(\lambda)_\nu - I_4) + \sum_{j>i} (\tilde{\mathcal{L}}_j(\lambda)_\nu - I_4)(\tilde{\mathcal{L}}_i(\lambda)_\nu - I_4) + \ldots \tag{3.6}$$

where $\tilde{\mathcal{L}}_j(\lambda)_\nu = V(\lambda)_\nu^{-j-1}\mathcal{L}_j(\lambda)_\nu V(\lambda)_\nu^j$ and I_4 denotes the 4×4 unit matrix.

For practical calculations, one should be careful that $V^{(2)}(\lambda,\mu)_\nu$ is not equal to the tensor product $V(\lambda)_\nu \otimes_{\mathrm{s}} V(\mu)_\nu$. There appear additional off-diagonal elements due to normal ordering of operators. Direct calculations lead us to the resulting forms for $V(\lambda)_\nu$ and $V^{(2)}(\lambda,\mu)_\nu$;

$$V(\lambda)_\nu = \mathrm{diag}(-\rho_8(\lambda,\nu), \rho_9(\lambda,\nu), \rho_9(\lambda,\nu), -\rho_1(\lambda,\nu)).$$

As for the matrix $V^{(2)}(\lambda,\mu)_\nu$, its diagonal consists of the elements of $V(\lambda)_\nu \otimes_{\mathrm{s}} V(\mu)_\nu$, and its non-vanishing off-diagonal elements are

$$\begin{aligned}
&V^{(2)}(\lambda,\mu)_\nu^{12,21} = V^{(2)}(\lambda,\mu)_\nu^{13,31} = -\mathrm{i}\rho_6(\lambda,\nu)\rho_6(\mu,\nu)\\
&V^{(2)}(\lambda,\mu)_\nu^{14,23} = -V^{(2)}(\lambda,\mu)_\nu^{14,32} = -\mathrm{i}\rho_6(\lambda,\nu)\rho_2(\mu,\nu)\\
&V^{(2)}(\lambda,\mu)_\nu^{24,42} = V^{(2)}(\lambda,\mu)_\nu^{34,43} = \mathrm{i}\rho_2(\lambda,\nu)\rho_2(\mu,\nu)\\
&V^{(2)}(\lambda,\mu)_\nu^{23,41} = -V^{(2)}(\lambda,\mu)_\nu^{32,41} = -\mathrm{i}\rho_2(\lambda,\nu)\rho_6(\mu,\nu)\\
&V^{(2)}(\lambda,\mu)_\nu^{14,41} = -\rho_3(\lambda,\nu)\rho_3(\mu,\nu).
\end{aligned}$$

Note that $V^{(2)}(\lambda,\mu)_\nu$ is upper triangular. Since the diagonals of $V^{(2)}(\lambda,\mu)_\nu$ and $V(\lambda) \otimes_{\mathrm{s}} V(\mu)_\nu$ are identical, $V^{(2)}(\lambda,\mu)_\nu$ can be diagonalized by an upper triangular matrix $U(\lambda,\mu)$ whose diagonal elements are all unity;

$$V^{(2)}(\lambda,\mu)_\nu = U(\lambda,\mu)_\nu (V(\lambda)_\nu \otimes_{\mathrm{s}} V(\mu)_\nu) U(\lambda,\mu)_\nu^{-1}. \tag{3.7}$$

Direct calculation leads us to a remarkable and surprising fact; $U(\lambda,\mu)_\nu$ is independent of ν. It is equal to $U(\lambda,\mu)_{\nu=0} = U(\lambda,\mu)$, which has appeared in the analysis of the usual Hubbard chain [20], so we will hereafter suppress the subscript ν in $U(\lambda,\mu)_\nu$. Its matrix elements are

$$\begin{aligned}
&U(\lambda,\mu)_{12,21} = U(\lambda,\mu)_{13,31} = -\mathrm{i}\rho_2/\rho_{10} \qquad U(\lambda,\mu)_{14,23} = -U(\lambda,\mu)_{14,32} = \mathrm{i}\rho_6/\rho_8\\
&U(\lambda,\mu)_{24,42} = U(\lambda,\mu)_{34,43} = \mathrm{i}\rho_2/\rho_9 \qquad U(\lambda,\mu)_{23,41} = -U(\lambda,\mu)_{32,41} = \mathrm{i}\rho_6/\rho_7\\
&U(\lambda,\mu)_{14,41} = -\rho_5/\rho_7
\end{aligned}$$

where $\rho_i = \rho_i(\lambda, \mu)$.

Taking the vacuum expectation value of the local exchange relation (2.3) yields

$$\check{\mathcal{R}}(\lambda, \mu) V^{(2)}(\lambda, \mu)_\nu = V^{(2)}(\mu, \lambda)_\nu \check{\mathcal{R}}(\lambda, \mu) \tag{3.8}$$

and we conclude that

$$\check{\mathcal{R}}(\lambda, \mu) \tilde{\mathcal{T}}^{(2)}(\lambda, \mu)_\nu = \tilde{\mathcal{T}}^{(2)}(\mu, \lambda)_\nu \check{\mathcal{R}}(\lambda, \mu). \tag{3.9}$$

Finally, collating (3.9) and other equations together, we arrive at the exchange relation for the monodromy matrix $\tilde{\mathcal{T}}(\lambda)_\nu$ on the infinite interval,

$$\tilde{\mathcal{R}}^{(+)}(\lambda, \mu)[\tilde{\mathcal{T}}(\lambda)_\nu \otimes_{\mathrm{s}} \tilde{\mathcal{T}}(\mu)_\nu] = [\tilde{\mathcal{T}}(\mu)_\nu \otimes_{\mathrm{s}} \tilde{\mathcal{T}}(\lambda)_\nu] \tilde{\mathcal{R}}^{(-)}(\lambda, \mu) \tag{3.10}$$

where

$$\tilde{\mathcal{R}}^{(\pm)}(\lambda, \mu)_\nu = U_\pm(\mu, \lambda)_\nu^{-1} \check{\mathcal{R}}(\lambda, \mu) U_\pm(\lambda, \mu)_\nu \tag{3.11}$$

$$U_\pm(\lambda, \mu)_\nu = \lim_{m \to \pm\infty} V^{(2)}(\lambda, \mu)_\nu^{-m} [V(\lambda)_\nu^m \otimes_{\mathrm{s}} V(\mu)_\nu^m]. \tag{3.12}$$

Since the calculation of the matrices $U_\pm(\lambda, \mu)_\nu$ and $\tilde{\mathcal{R}}^{(\pm)}(\lambda, \mu)_\nu$ is rather technical, we do not reproduce it here. Its details are presented in [21] in the case of $\nu = 0$ (Hubbard model). The only point we should note here is that apart from some singular points (e.g. $\lambda = \mu$), we can say that $U(\lambda, \mu) = U_\pm(\lambda, \mu)_\nu$ and

$$\tilde{\mathcal{R}}(\lambda, \mu) = \tilde{\mathcal{R}}^{(\pm)}(\lambda, \mu)_\nu$$

$$= \begin{pmatrix}
\rho_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\rho_1\rho_4}{\mathrm{i}\rho_{10}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\rho_1\rho_4}{\mathrm{i}\rho_{10}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-\rho_1\rho_4}{\rho_5-\rho_4} & 0 & 0 & 0 \\
0 & -\mathrm{i}\rho_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\rho_3\rho_4-\rho_2^2}{\rho_3-\rho_1} & 0 & 0 & \frac{\rho_9\rho_{10}}{\rho_3-\rho_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{i}\rho_1\rho_4}{\rho_9} & 0 & 0 \\
0 & 0 & -\mathrm{i}\rho_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\rho_9\rho_{10}}{\rho_3-\rho_1} & 0 & 0 & \frac{\rho_3\rho_4-\rho_2^2}{\rho_3-\rho_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{i}\rho_1\rho_4}{\rho_9} & 0 \\
0 & 0 & 0 & \rho_1-\rho_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}\rho_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}\rho_9 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_1
\end{pmatrix} \tag{3.13}$$

where $\rho_i = \rho_i(\lambda, \mu)$. Let us write the elements of $\tilde{\mathcal{T}}(\lambda)$ as

$$\tilde{\mathcal{T}}(\lambda)_\nu = \begin{pmatrix}
D_{11}(\lambda)_\nu & C_{11}(\lambda)_\nu & C_{12}(\lambda)_\nu & D_{12}(\lambda)_\nu \\
B_{11}(\lambda)_\nu & A_{11}(\lambda)_\nu & A_{12}(\lambda)_\nu & B_{12}(\lambda)_\nu \\
B_{21}(\lambda)_\nu & A_{21}(\lambda)_\nu & A_{22}(\lambda)_\nu & B_{22}(\lambda)_\nu \\
D_{21}(\lambda)_\nu & C_{21}(\lambda)_\nu & C_{22}(\lambda)_\nu & D_{22}(\lambda)_\nu
\end{pmatrix}. \tag{3.14}$$

Since the $\tilde{\mathcal{R}}$-matrix is independent of the value of ν, the commutation rules between the elements of $\tilde{\mathcal{T}}(\lambda)$, which are obtained from the exchange relation (3.10), are exactly the same as those in the $\nu = 0$ case, i.e. the usual Hubbard model. A complete list of the commutation rules is found in appendix B of [21], and it is the same in this case.

4. Yangian symmetry and commuting operators

If we follow the notion of the quantum inverse scattering method, the remaining task is to investigate the meaning of each matrix element of $\tilde{\mathcal{T}}(\lambda)_\nu$. As is also the case for the

Hubbard model [21], the commutation relations between the elements of the submatrix $A(\lambda)_\nu$ decouple from the rest of the algebra

$$r(\lambda,\mu)\,(A(\lambda)_\nu \otimes A(\mu)_\nu) = (A(\mu)_\nu \otimes A(\lambda)_\nu)\,r(\lambda,\mu) \tag{4.1}$$

where

$$r(\lambda,\mu) = \frac{\rho_3\rho_4 - \rho_2^2 + \rho_9\rho_{10}\mathcal{P}}{\rho_4(\rho_3-\rho_1)} \qquad (\rho_j = \rho_j(\lambda,\mu)) \tag{4.2}$$

and $\mathcal{P}$ is a 4×4 permutation matrix ($\mathcal{P}x\otimes y = y\otimes x$). As is remarked previously, (4.1) is identical to the one in the Hubbard model ($\nu = 0$) and we can follow the same argument as in the previous work [21]. By the reparametrization

$$v(\lambda) = -2\mathrm{i}\cot 2\lambda \cosh 2h(\lambda) \tag{4.3}$$

the R-matrix $r(\lambda,\mu)$ turns into the rational R-matrix of the XXX spin chain,

$$r(\lambda,\mu) = \frac{\mathrm{i}U + (v(\lambda)-v(\mu))\mathcal{P}}{\mathrm{i}U + v(\lambda)-v(\mu)}. \tag{4.4}$$

Let us expand $A(\lambda)_\nu$ in terms of $v(\lambda)^{-1}$,

$$A(\lambda)_\nu = I_2 + \mathrm{i}U\sum_{n=0}^{\infty}\frac{1}{v(\lambda)^{n+1}}\left(\sum_{a=1}^{3} Q_n^a(\nu)\tilde{\sigma}^a + Q_n^0(\nu)I_2\right) \tag{4.5}$$

where $\tilde{\sigma}^x = -\sigma^y$, $\tilde{\sigma}^y = \sigma^x$, $\tilde{\sigma}^z = \sigma^z$, and I_2 is the 2×2 unit matrix. This expansion can be achieved by considering the limit $v(\lambda)\to\infty$ as $\mathrm{Im}\,(\lambda)\to\infty$ and by choosing the proper branch of the solution of (2.5), which determines h as a function of λ. Equation (2.5) implies that

$$\mathrm{e}^{-2h(\lambda)} = -\frac{U}{4}\sin 2\lambda \pm \sqrt{1+\left(\frac{U}{4}\sin 2\lambda\right)^2}. \tag{4.6}$$

To achieve convergence of the matrix elements $A(\lambda)_\nu$, we have to choose the lower sign here. Then it follows from general considerations [24–26] that the first six operators $Q_0^a(\nu)$, $Q_1^a(\nu)$ generate a representation of the Y(su(2)) Yangian quantum group.

There is an alternative description of the Yangian Y(su(2)) [28] described below. The Yangian Y(su(2)) is a Hopf algebra spanned by six generators Q_n^a ($n = 0, 1,\ a = x, y, z$), satisfying the following relations,

$$[Q_0^a, Q_0^b] = f^{abc}Q_0^c \tag{4.7}$$

$$[Q_0^a, Q_1^b] = f^{abc}Q_1^c \tag{4.8}$$

$$\begin{aligned}&[[Q_1^a, Q_1^b],[Q_0^c, Q_1^d]] + [[Q_1^c, Q_1^d],[Q_0^a, Q_1^b]]\\ &\qquad = \kappa^2(A^{abkefg}f^{cdk} + A^{cdkefg}f^{abk})\{Q_0^e, Q_0^f, Q_1^g\}.\end{aligned} \tag{4.9}$$

Here κ is a nonzero constant, $f^{abc} = \mathrm{i}\varepsilon^{abc}$ is the antisymmetric tensor of structure constants of su(2), and $A^{abcdef} = f^{adk}f^{bel}f^{cfm}f^{klm}$. The bracket { } in (4.9) denotes the symmetrized product

$$\{x_1, x_2, x_3\} = \frac{1}{3!}\sum_{\sigma\in S_3} x_{\sigma 1}x_{\sigma 2}x_{\sigma 3}. \tag{4.10}$$

The Hopf algebra structure of Y(su(2)) is described in [28] and its representation theory, which will be used later, is developed in [29, 30].

We can use (4.5) and (3.6) to obtain the representation of Yangian generators;

$$Q_0^a(\nu) = \frac{1}{2}\sum_j \sigma^a_{\alpha\beta} c^\dagger_{j,\alpha} c_{j,\beta} \tag{4.11}$$

$$\begin{aligned} Q_1^a(\nu) = \frac{\mathrm{i}}{2\sin\nu\cos\nu}\Bigg[&\sum_{i>j} (\tan\nu)^{i-j} \mathrm{e}^{2h(\nu)(2-n_i-n_j)} \sigma^a_{\alpha\beta} c^\dagger_{i\alpha} c_{j\beta} \\ &+ \sum_{i<j} (-\cot\nu)^{i-j} \mathrm{e}^{-2h(\nu)(2-n_i-n_j)} \sigma^a_{\alpha\beta} c^\dagger_{i\alpha} c_{j\beta} \Bigg] \\ &- \frac{\mathrm{i}U}{4} \sum_{i,j} \mathrm{sgn}(j-i) \sigma^a_{\alpha\beta} c^\dagger_{i,\alpha} c^\dagger_{j,\gamma} c_{i,\gamma} c_{j,\beta} + \mathrm{i}U \cot 2h(\nu)\ \sin^2\nu\ Q_0^a(\nu). \end{aligned} \tag{4.12}$$

In this case the constant κ in (4.9) is equal to iU. Note that $Q_0^a(\nu) = S^a$ is just the operator of the a-component of the total spin. The representation of the Yangian algebra in the usual Hubbard model [10, 20] is a special case of $\nu = 0$ in (4.11) and (4.12).

Since the quantum determinant

$$\mathrm{Det}_q A(\lambda)_\nu = A_{11}(\lambda)_\nu A_{22}(\check{\lambda})_\nu - A_{12}(\lambda)_\nu A_{21}(\check{\lambda})_\nu \tag{4.13}$$

where $\check{\lambda}$ is defined by the relation $v(\check{\lambda}) = v(\lambda) - \mathrm{i}U$, is in the centre of the Yangian

$$[\mathrm{Det}_q A(\lambda)_\nu, A(\mu)_\nu] = 0 \tag{4.14}$$

it provides a generating function of mutually commuting operators,

$$[\mathrm{Det}_q A(\lambda)_\nu, \mathrm{Det}_q A(\mu)_\nu] = 0. \tag{4.15}$$

The asymptotic expansion in terms of $v(\lambda)^{-1}$,

$$\mathrm{Det}_q A(\lambda)_\nu = 1 + \mathrm{i}U \sum_{n=0}^{\infty} \frac{J_n(\nu)}{v(\lambda)^{n+1}} \tag{4.16}$$

produces $J_0(\nu) = 0$, $J_1(\nu) = \mathrm{i}\hat{H}_{\mathrm{long}}$, where

$$\begin{aligned} \hat{H}_{\mathrm{long}} = -\frac{1}{\sin\nu\cos\nu}\Bigg[&\sum_{i>j} (\tan\nu)^{i-j} \mathrm{e}^{2h(\nu)(1-n_{i,-\sigma}-n_{j,-\sigma})} c^\dagger_{i\sigma} c_{j\sigma} \\ &- \sum_{i<j} (-\cot\nu)^{i-j} \mathrm{e}^{-2h(\nu)(1-n_{i,-\sigma}-n_{j,-\sigma})} c^\dagger_{i\sigma} c_{j\sigma} \Bigg] \\ &+ U(1+2\sin^2\nu) \sum_i [(n_{j\uparrow} - \tfrac{1}{2})(n_{j\downarrow} - \tfrac{1}{2}) - \tfrac{1}{4}]. \end{aligned} \tag{4.17}$$

Due to the relation (4.15), the $J_n(\nu)$'s mutually commute. Therefore, $\hat{H}_{\mathrm{long}}$ can be embedded in a family of infinite number of commuting operators, and can be regarded as a integrable Hamiltonian. Moreover, (4.14) indicates the Y(su(2)) invariance of the model;

$$[Q_0^a(\nu), \hat{H}_{\mathrm{long}}] = 0 = [Q_1^a(\nu), \hat{H}_{\mathrm{long}}] \qquad (a = 1, 2, 3). \tag{4.18}$$

In particular, it implies that the model is su(2) invariant. By subtracting a constant from $\hat{H}_{\mathrm{long}}$, we can make this Hamiltonian invariant under partial particle–hole transformation (2.2):

$$\hat{H}'_{\mathrm{long}} = -\frac{1}{\sin\nu\cos\nu}\Bigg[\sum_{i>j} (\tan\nu)^{i-j} \mathrm{e}^{2h(\nu)(1-n_{i,-\sigma}-n_{j,-\sigma})} c^\dagger_{i\sigma} c_{j\sigma}$$

$$
-\sum_{i<j}(-\cot\nu)^{i-j}\,\mathrm{e}^{-2h(\nu)(1-n_{i,-\sigma}-n_{j,-\sigma})}c^{\dagger}_{i\sigma}c_{j\sigma}\Bigg]
+U(1+2\sin^2\nu)\sum_j(n_{j\uparrow}-\tfrac{1}{2})(n_{j\downarrow}-\tfrac{1}{2}). \tag{4.19}
$$

This complicated Hamiltonian can be made simpler by noting (2.5) to obtain

$$
\hat{H}''_{\text{long}} = -\mathrm{i}\sin\nu\cos\nu\,\hat{H}'_{\text{long}} = \mathrm{i}\sum_{i>j}\left(\frac{\mathrm{i}}{r}\right)^{i-j}\mathrm{e}^{\mathrm{i}J(1-n_{i,-\sigma}-n_{j,-\sigma})}c^{\dagger}_{i\sigma}c_{j\sigma}+\text{h.c.}
+\frac{6-2r^2}{1-r^2}\sin J\sum_j(n_{j\uparrow}-\tfrac{1}{2})(n_{j\downarrow}-\tfrac{1}{2}) \tag{4.20}
$$

where $r=\mathrm{i}\cot\nu$, $J=-2\mathrm{i}h(\nu)$ and r and J are real. We will, however, use the Hamiltonian (4.19) instead of (4.20), since it is easier to extract information about the model out of the quantum inverse scattering method.

The Y(su(2)) invariance of $\hat{H}'_{\text{long}}$ shown in (4.18), together with the invariance under the partial particle–hole transformation (2.2), leads us to the result

$$
[Q^a_0(\nu),\hat{H}'_{\text{long}}]=0=[Q^a_1(\nu),\hat{H}'_{\text{long}}] \tag{4.21}
$$

$$
[Q^{a\prime}_0(\nu),\hat{H}'_{\text{long}}]=0=[Q^{a\prime}_1(\nu),\hat{H}'_{\text{long}}] \tag{4.22}
$$

where $Q^{a\prime}_n(\nu)$ is obtained by performing the partial particle–hole transformation (2.2) to$Q^a_n(\nu)$. By construction, the operators $Q^{a\prime}_n(\nu)$ form another Y(su(2)) algebra, and we can straightforwardly verify that $[Q^a_m(\nu), Q^{b\prime}_n(\nu)]$ vanishes for $a,b=x,y,z;\ m,n=0,1$. Therefore, we can say that the Hamiltonian $\hat{H}'_{\text{long}}$ is $\mathrm{Y(su(2))}\oplus\mathrm{Y(su(2))}$ invariant.

Hereafter we shall assume $\hat{H}'_{\text{long}}$ to be Hermitian, i.e. U is real and both ν and $h(\nu)$ are pure imaginary. With this assumption, $\hat{H}'_{\text{long}}$ can be regarded as a new Hamiltonian for electrons with on-site interaction and variable range hopping. The amplitude of the hopping decays exponentially with the hopping range. There is also an interference effect due to the term $\exp(\pm 2h(\nu)(1-n_{i,-\sigma}-n_{j,-\sigma}))$. One can easily check that in the $\nu=0$ limit the hopping terms vanish except for the ones to the neighbouring sites, and the usual Hubbard model is restored in this limit. In that sense it is an integrable extension of the Hubbard model (2.1) with variable range hopping.

There is another integrable Hubbard model with variable range hopping discovered earlier [31]. Its Hamiltonian is given by

$$
H=\sum_{\sigma,i\neq j}t(i-j)c^{\dagger}_{i\sigma}c_{j\sigma}+U\sum_j n_{j\uparrow}n_{j\downarrow} \tag{4.23}
$$

with

$$
t(s)=-\mathrm{i}t(-1)^s\left(\frac{L}{\pi}\sin\frac{\pi s}{L}\right)^{-1} \tag{4.24}
$$

or

$$
t(s)=-\mathrm{i}\sinh\kappa(-1)^s/\sinh(\kappa s). \tag{4.25}
$$

Although both our model and the above model contain the usual nearest-neighbour-hopping Hubbard model as a limiting case, we do not know whether they can be related to each other.

To close this section, we shall add a comment. The Hamiltonian $\hat{H}_{\text{long}}$ obtained here is different from $\hat{H}_\nu$ obtained from a logarithmic derivative of the monodromy matrix $\mathcal{T}_{mn}(\lambda)_\nu$

on the finite interval. Such things do not occur in the previously studied cases of the fermionic nonlinear Schrödinger model [26] or the Hubbard model [20, 21]. The relation between the two Hamiltonians $\hat{H}_{\rm long}$ and $\hat{H}_\nu$ is left as a future problem.

5. Construction of eigenvectors

5.1. Creation operators of quasiparticles

As is the case with the usual Hubbard chain and other integrable models, the entries of the monodromy matrix $\tilde{\mathcal{T}}(\lambda)$ can be used to construct multiparticle eigenstates on the vacuum. By calculating commutators between these entries of $\tilde{\mathcal{T}}(\lambda)$ and the particle number operator $\hat{N}$, we can see that $B_{a1}(\lambda)_\nu$ and $C_{2a}(\lambda)_\nu$ add $\hat{N}$ by one and $D_{21}(\lambda)_\nu$ adds $\hat{N}$ by two, while $B_{a2}(\lambda)_\nu$ and $C_{1a}(\lambda)_\nu$ reduce $\hat{N}$ by one and $D_{12}(\lambda)_\nu$ does by two. $A_{ab}(\lambda)_\nu$, $D_{11}(\lambda)_\nu$ and $D_{22}(\lambda)_\nu$ keep $\hat{N}$ unchanged. To gain further insight, let us calculate actions of the operators in $\tilde{\mathcal{T}}(\lambda)$ onto the vacuum $|0\rangle$. By using (3.6), some simplest ones are calculated as follows

$$B_{11}(\lambda)_\nu|0\rangle = -\frac{\mathrm{i}\rho_6}{\rho_9}\sum_j \mathrm{e}^{-\mathrm{i}jp(\lambda,\nu)}c^\dagger_{j\downarrow}|0\rangle \tag{5.1}$$

$$B_{21}(\lambda)_\nu|0\rangle = \frac{\rho_6}{\rho_9}\sum_j \mathrm{e}^{-\mathrm{i}jp(\lambda,\nu)}c^\dagger_{j\uparrow}|0\rangle \tag{5.2}$$

$$C_{21}(\lambda)_\nu|0\rangle = -\frac{\rho_2}{\rho_1}\sum_j \mathrm{e}^{-\mathrm{i}jk(\lambda,\nu)}c^\dagger_{j\uparrow}|0\rangle \tag{5.3}$$

$$C_{22}(\lambda)_\nu|0\rangle = -\frac{\mathrm{i}\rho_2}{\rho_1}\sum_j \mathrm{e}^{-\mathrm{i}jk(\lambda,\nu)}c^\dagger_{j\downarrow}|0\rangle \tag{5.4}$$

$$\begin{aligned} D_{21}(\lambda)_\nu|0\rangle = \sum_{j,l} c^\dagger_{j\uparrow}c^\dagger_{l\downarrow}\Bigg[&\theta(j>l)\frac{\mathrm{i}\rho_6\rho_2}{\rho_9\rho_1}\mathrm{e}^{-\mathrm{i}jk(\lambda,\nu)-\mathrm{i}lp(\lambda,\nu)} + \theta(j<l)\frac{\mathrm{i}\rho_6\rho_2}{\rho_9\rho_1}\mathrm{e}^{-\mathrm{i}jp(\lambda,\nu)-\mathrm{i}lk(\lambda,\nu)} \\ &+ \delta_{jl}\frac{\mathrm{i}\rho_3}{\rho_1}\mathrm{e}^{-\mathrm{i}j\{p(\lambda,\nu)+k(\lambda,\nu)\}}\Bigg]|0\rangle \end{aligned} \tag{5.5}$$

where $\rho_j = \rho_j(\lambda,\nu)$ and

$$\mathrm{e}^{-\mathrm{i}k(\lambda,\nu)} = -\rho_9(\lambda,\nu)/\rho_1(\lambda,\nu) \qquad \mathrm{e}^{-\mathrm{i}p(\lambda,\nu)} = -\rho_8(\lambda,\nu)/\rho_9(\lambda,\nu). \tag{5.6}$$

The commutators between $\mathrm{Det}_q(A(\mu)_\nu)$ and the various operators in $\tilde{\mathcal{T}}(\lambda)_\nu$ are calculated from the exchange relation (3.10). The resulting commutators are the same as the case of the original Hubbard model ($\nu = 0$), which is summarized in appendix B.2 of [21]. Hence, the commutators between the Hamiltonian $\hat{H}_{\rm long}$ and the operators in the matrix $\tilde{\mathcal{T}}(\lambda)_\nu$ are also the same as the $\nu = 0$ case;

$$[\hat{H}_{\rm long}, B_{a1}(\lambda)_\nu] = -(2\cos p(\lambda) + U/2)B_{a1}(\lambda)_\nu \tag{5.7}$$

$$[\hat{H}_{\rm long}, B_{a2}(\lambda)_\nu] = (2\cos k(\lambda) + U/2)B_{a2}(\lambda)_\nu \tag{5.8}$$

$$[\hat{H}_{\rm long}, C_{1a}(\lambda)_\nu] = (2\cos p(\lambda) + U/2)C_{1a}(\lambda)_\nu \tag{5.9}$$

$$[\hat{H}_{\rm long}, C_{2a}(\lambda)_\nu] = -(2\cos k(\lambda) + U/2)C_{2a}(\lambda)_\nu \tag{5.10}$$

$$[\hat{H}_{\rm long}, D_{12}(\lambda)_\nu] = 2(\mathrm{e}^{\mathrm{i}p(\lambda)} + \mathrm{e}^{-\mathrm{i}k(\lambda)})D_{12}(\lambda)_\nu \tag{5.11}$$

$$[\hat{H}_{\rm long}, D_{21}(\lambda)_\nu] = -2(\mathrm{e}^{\mathrm{i}p(\lambda)} + \mathrm{e}^{-\mathrm{i}k(\lambda)})D_{21}(\lambda)_\nu \tag{5.12}$$

where

$$\mathrm{e}^{\mathrm{i}k(\lambda)} = -\mathrm{e}^{2h(\lambda)}\cot\lambda \qquad \mathrm{e}^{\mathrm{i}p(\lambda)} = -\mathrm{e}^{-2h(\lambda)}\cot\lambda. \tag{5.13}$$

Other entries $D_{11}(\lambda)$, $D_{22}(\lambda)$ and $A_{ab}(\lambda)$ commute with $\hat{H}_{\mathrm{long}}$. The above results justify the interpretation of $B_{a1}(\lambda)$,$C_{2a}(\lambda)$ and $D_{21}(\lambda)$ as creation operators. $B_{a1}(\lambda)$ and $C_{2a}(\lambda)$ create single-particle excitations, whereas $D_{21}(\lambda)$ creates a bound state of two particles. For example, from $\hat{H}_{\mathrm{long}}|0\rangle = 0$ and (5.10) we deduce

$$\hat{H}_{\mathrm{long}} C_{2a_1}(\lambda_1)_\nu \ldots C_{2a_n}(\lambda_n)_\nu|0\rangle = -\sum_{j=1}^{n}(2\cos k(\lambda_j) + U/2)\, C_{2a_1}(\lambda_1)_\nu \ldots C_{2a_n}(\lambda_n)_\nu|0\rangle. \tag{5.14}$$

Similarly, the applications of the operators $B_{a1}(\lambda)$ or mixed products of operators $B_{a1}(\lambda)$ and $C_{2a}(\lambda)$ on the vacuum produce eigenstates of the Hamiltonian.

Let us consider the relation between these creation operators and Yangian Y(su(2)). Since the Yangian generators $Q_n^a(\nu)$ ($n = 0, 1$; $a = x, y, z$) are coefficients of the power expansion of $A(\mu)_\nu$, the commutators between $Q_n^a(\nu)$ and operators $B(\lambda)_\nu$, $C(\lambda)_\nu$, and $D(\lambda)_\nu$ can be obtained from (3.10). The results are the same as in the case of the Hubbard model [21];

$$[Q_0^a(\nu), B(\lambda)_\nu] = -\tfrac{1}{2}\tilde{\sigma}^a B(\lambda)_\nu \tag{5.15}$$

$$[Q_1^a(\nu), B(\lambda)_\nu] = \sin p(\lambda)\tilde{\sigma}^a B(\lambda)_\nu + \frac{U}{2}\varepsilon^{abc}\tilde{\sigma}^b B(\lambda)_\nu Q_0^c(\nu) \tag{5.16}$$

$$[Q_0^a(\nu), C(\lambda)_\nu] = \tfrac{1}{2}C(\lambda)_\nu\tilde{\sigma}^a \tag{5.17}$$

$$[Q_1^a(\nu), C(\lambda)_\nu] = -\sin k(\lambda)C(\lambda)_\nu\tilde{\sigma}^a + \frac{U}{2}\varepsilon^{abc}C(\lambda)_\nu\tilde{\sigma}^b Q_0^c(\nu) \tag{5.18}$$

$$[Q_0^a(\nu), D(\lambda)_\nu] = [Q_1^a(\nu), D(\lambda)_\nu] = 0. \tag{5.19}$$

These commutators will be used to investigate Yangian representations of the multiparticle eigenstates.

5.2. Scattering states

Observing the cases of other integrable models studied earlier [22, 32], we propose the following two pairs of normalized creation operators of scattering states,

$$R_\alpha(\lambda)_\nu^\dagger = \mathrm{i}^{3-\alpha}\frac{\rho_1(\lambda,\nu)}{\rho_2(\lambda,\nu)}\, C_{2\alpha}(\lambda)_\nu D_{22}(\lambda)_\nu^{-1} \qquad (\alpha = 1, 2) \tag{5.20}$$

$$\hat{R}_\alpha(\lambda)_\nu^\dagger = \mathrm{i}^{\alpha-1}\frac{\rho_9(\lambda,\nu)}{\rho_6(\lambda,\nu)} B_{3-\alpha,1}(\lambda)_\nu D_{11}(\lambda)_\nu^{-1} \qquad (\alpha = 1, 2). \tag{5.21}$$

In these formulae $\alpha = 1$ corresponds to spin-up and $\alpha = 2$ to spin-down. The numerical prefactors have been obtained by demanding that $R_\alpha(\lambda)_\nu^\dagger$ and $\hat{R}_\alpha(\lambda)_\nu^\dagger$ generate normalized one-particle states,

$$R_\alpha(\lambda)_\nu^\dagger|0\rangle = \sum_j \mathrm{e}^{-\mathrm{i}jk(\lambda,\nu)}c_{j,\alpha}^\dagger|0\rangle \qquad \hat{R}_\alpha(\lambda)_\nu^\dagger|0\rangle = \sum_j \mathrm{e}^{-\mathrm{i}jp(\lambda,\nu)}c_{j,\alpha}^\dagger|0\rangle. \tag{5.22}$$

Hereafter we assume that λ is chosen in such a way that $R_\alpha(\lambda)_\nu^\dagger$ and $\hat{R}_\alpha(\lambda)_\nu^\dagger$ create physical states. This means for $R_\alpha(\lambda)_\nu^\dagger$ that $k(\lambda,\nu)$ has to be real and for $\hat{R}_\alpha(\lambda)_\nu^\dagger$ that $p(\lambda,\nu)$ has to be real.

By the method in [33], Hermitian conjugation can be performed on the operators $R_\alpha(\lambda)_\nu$ and $\hat{R}_\alpha(\lambda)_\nu$, and the resulting normalized annihilation operators are

$$R_\alpha(\lambda)_\nu = \mathrm{i}^{2-\alpha}\frac{\rho_8(\lambda',\nu)}{\rho_6(\lambda',\nu)} D_{11}(\lambda)_\nu^{-1} C_{1,3-\alpha}(\lambda)_\nu \tag{5.23}$$

$$\hat{R}_\alpha(\lambda)_\nu = \mathrm{i}^{\alpha-2}\frac{\rho_9(\lambda',\nu)}{\rho_2(\lambda',\nu)} D_{22}(\lambda)_\nu^{-1} B_{\alpha 2}(\lambda)_\nu \tag{5.24}$$

with $\lambda' = \pi/2 - \lambda^*$. The commutation rules between the operators $R_\alpha(\lambda)_\nu^\dagger$, $\hat{R}_\alpha(\lambda)_\nu^\dagger$, $R_\alpha(\lambda)_\nu$, $\hat{R}_\alpha(\lambda)_\nu$ are

$$R_\alpha(\lambda)_\nu^\dagger R_\beta(\mu)_\nu^\dagger = -r(\lambda,\mu)_{\gamma\delta,\alpha\beta} R_\gamma(\mu)_\nu^\dagger R_\delta(\lambda)_\nu^\dagger \tag{5.25}$$
$$R_\alpha(\lambda)_\nu R_\beta(\mu)_\nu^\dagger = -r(\mu,\lambda)_{\gamma\alpha,\delta\beta} R_\gamma(\mu)_\nu^\dagger R_\delta(\lambda)_\nu \tag{5.26}$$
$$\hat{R}_\alpha(\lambda)_\nu^\dagger \hat{R}_\beta(\mu)_\nu^\dagger = -r(\mu,\lambda)_{\gamma\delta,\alpha\beta} \hat{R}_\gamma(\mu)_\nu^\dagger \hat{R}_\delta(\lambda)_\nu^\dagger \tag{5.27}$$
$$\hat{R}_\alpha(\lambda)_\nu \hat{R}_\beta(\mu)_\nu^\dagger = -r(\lambda,\mu)_{\gamma\alpha,\delta\beta} \hat{R}_\gamma(\mu)_\nu^\dagger \hat{R}_\delta(\lambda)_\nu \tag{5.28}$$
$$R_\alpha(\lambda)_\nu^\dagger \hat{R}_\beta(\mu)_\nu^\dagger = -\hat{R}_\beta(\mu)_\nu^\dagger R_\alpha(\lambda)_\nu^\dagger \tag{5.29}$$
$$R_\alpha(\lambda)_\nu \hat{R}_\beta(\mu)_\nu^\dagger = -\hat{R}_\beta(\mu)_\nu^\dagger R_\alpha(\lambda)_\nu. \tag{5.30}$$

Here we neglected δ-function contributions from some singular points, e.g. $\lambda = \mu$.

The operators $R_\alpha(\lambda)$, $R_\alpha(\lambda)^\dagger$ and $\hat{R}_\alpha(\lambda)$, $\hat{R}_\alpha(\lambda)^\dagger$ form a representation of the graded Zamolodchikov–Faddeev algebra with S-matrix $r(\lambda,\mu)$. These representations may be identified as representations of left and right Zamolodchikov–Faddeev algebra, respectively [34, 22, 35–37]. The operators $R_\alpha(\lambda)^\dagger$ and $\hat{R}_\alpha(\lambda)^\dagger$ are graded as odd, which implies that they are creation operators of fermionic quasiparticles.

We shall present two-particle states generated by $R_\alpha(\lambda)^\dagger$ to know the physical meanings of $R_\alpha^\dagger(\lambda)_\nu$;

$$R_1(\lambda)_\nu^\dagger R_1(\mu)_\nu^\dagger|0\rangle = \sum_{j,l} c_{j\uparrow}^\dagger c_{l\uparrow}^\dagger \mathrm{e}^{-\mathrm{i}jk(\lambda,\nu)}\mathrm{e}^{-\mathrm{i}lk(\mu,\nu)}|0\rangle \tag{5.31}$$

$$R_2(\lambda)_\nu^\dagger R_2(\mu)_\nu^\dagger|0\rangle = \sum_{j,l} c_{j\downarrow}^\dagger c_{l\downarrow}^\dagger \mathrm{e}^{-\mathrm{i}jk(\lambda,\nu)}\mathrm{e}^{-\mathrm{i}lk(\mu,\nu)}|0\rangle \tag{5.32}$$

$$\begin{aligned} R_1(\lambda)_\nu^\dagger R_2(\mu)_\nu^\dagger|0\rangle = \sum_{j,l} c_{j\uparrow}^\dagger c_{l\downarrow}^\dagger \Bigg[&\theta(j<l)\mathrm{e}^{-\mathrm{i}jk(\lambda,\nu)}\mathrm{e}^{-\mathrm{i}lk(\mu,\nu)}\frac{v(\lambda)-v(\mu)}{v(\lambda)-v(\mu)+\mathrm{i}U} \\ &+\theta(j<l)\mathrm{e}^{-\mathrm{i}jk(\lambda,\nu)}\mathrm{e}^{-\mathrm{i}lk(\mu,\nu)} + \theta(j<l)\mathrm{e}^{-\mathrm{i}lk(\lambda,\nu)}\mathrm{e}^{-\mathrm{i}jk(\mu,\nu)}\frac{-\mathrm{i}U}{v(\lambda)-v(\mu)+\mathrm{i}U} \\ &+\delta_{jl}\mathrm{e}^{-\mathrm{i}j\{k(\lambda,\nu)+k(\mu,\nu)\}}F(\lambda,\mu,\nu)\Bigg]|0\rangle \end{aligned} \tag{5.33}$$

$$\begin{aligned} R_2(\lambda)_\nu^\dagger R_1(\mu)_\nu^\dagger|0\rangle = \sum_{j,l} c_{j\downarrow}^\dagger c_{l\uparrow}^\dagger \Bigg[&\theta(j>l)\mathrm{e}^{-\mathrm{i}jk(\lambda,\nu)}\mathrm{e}^{-\mathrm{i}lk(\mu,\nu)}\frac{v(\lambda)-v(\mu)}{v(\lambda)-v(\mu)+\mathrm{i}U} \\ &+\theta(j<l)\mathrm{e}^{-\mathrm{i}jk(\lambda,\nu)}\mathrm{e}^{-\mathrm{i}lk(\mu,\nu)} + \theta(j<l)\mathrm{e}^{-\mathrm{i}lk(\lambda,\nu)}\mathrm{e}^{-\mathrm{i}jk(\mu,\nu)}\frac{-\mathrm{i}U}{v(\lambda)-v(\mu)+\mathrm{i}U} \\ &+\delta_{jl}\mathrm{e}^{-\mathrm{i}j\{k(\lambda,\nu)+k(\mu,\nu)\}}F(\lambda,\mu,\nu)\Bigg]|0\rangle \end{aligned} \tag{5.34}$$

where

$$F(\lambda,\mu,\nu) = \frac{\rho_9\rho_6}{\rho_4\rho_8}(\lambda,\mu)(\mathrm{e}^{h(\lambda)+h(\mu)-2h(\nu)}\cos\lambda\cos\mu + \mathrm{e}^{-h(\lambda)-h(\mu)+2h(\nu)}\sin\lambda\sin\mu). \tag{5.35}$$

From these wavefunctions, similar to the case of the Hubbard model, we conjecture that the n-particle state

$$R_{\alpha_1}(\lambda_1)_\nu^\dagger \ldots R_{\alpha_n}(\lambda_n)_\nu^\dagger|0\rangle \tag{5.36}$$

is a normalized in-state if $k(\lambda_1, \nu) < \cdots < k(\lambda_n, \nu)$ and a normalized out-state if $k(\lambda_1, \nu) > \cdots > k(\lambda_n, \nu)$. Here 'normalized' means that the magnitude of the incident wave is unity. Similarly, for the $\hat{R}$ operators, we conjecture that the n-particle state

$$\hat{R}_{\alpha_1}(\lambda_1)^\dagger_\nu \ldots \hat{R}_{\alpha_n}(\lambda_n)^\dagger_\nu |0\rangle \tag{5.37}$$

is a normalized out-state if $p(\lambda_1, \nu) < \cdots < p(\lambda_n, \nu)$ and a normalized in-state if $p(\lambda_1, \nu) > \cdots > p(\lambda_n, \nu)$. We have not discovered its proof yet.

The Yangian representation of the multiparticle states can be investigated by the following commutators from (5.15)–(5.19);

$$[Q_0^a(\nu), R_\alpha(\lambda)^\dagger_\nu] = \tfrac{1}{2} R_\beta(\lambda)^\dagger_\nu \sigma^a_{\beta\alpha} \tag{5.38}$$

$$[Q_1^a(\nu), R_\alpha(\lambda)^\dagger_\nu] = -\sin k(\lambda) R_\beta(\lambda)^\dagger_\nu \sigma^a_{\beta\alpha} + \frac{U}{2} \varepsilon^{abc} R_\beta(\lambda)^\dagger_\nu \sigma^b_{\beta\alpha} Q_0^c(\nu) \tag{5.39}$$

$$[Q_0^a(\nu), \hat{R}_\alpha(\lambda)^\dagger_\nu] = \tfrac{1}{2} \hat{R}_\beta(\lambda)^\dagger_\nu \sigma^a_{\beta\alpha} \tag{5.40}$$

$$[Q_1^a(\nu), \hat{R}_\alpha(\lambda)^\dagger_\nu] = -\sin p(\lambda) \hat{R}_\beta(\lambda)^\dagger_\nu \sigma^a_{\beta\alpha} - \frac{U}{2} \varepsilon^{abc} \hat{R}_\beta(\lambda)^\dagger_\nu \sigma^b_{\beta\alpha} Q_0^c(\nu). \tag{5.41}$$

These formulae induce an action of the Yangian on n-particle states [38, 26, 21]. Noting that $Q_0^a(\nu)|0\rangle = 0 = Q_1^a(\nu)|0\rangle$, we obtain the action of the Yangian on the $n = 1$ sector as

$$Q_0^a(\nu) R_\alpha(\lambda)^\dagger_\nu |0\rangle = \tfrac{1}{2} \sigma^a_{\beta\alpha} R_\beta(\lambda)^\dagger |0\rangle \tag{5.42}$$

$$Q_1^a(\nu) R_\alpha(\lambda)^\dagger_\nu |0\rangle = -\sin k(\lambda) \sigma^a_{\beta\alpha} R_\beta(\lambda)^\dagger_\nu |0\rangle \tag{5.43}$$

which is identified as the fundamental representation $W_1(-2\sin k(\lambda))$.

The 2^n-dimensional representation formed by n-particle states (5.36) can be studied in a similar manner as [26, 21], and is identified as the tensor product representation $W_1(-2\sin k(\lambda_1)) \otimes \ldots \otimes W_1(-2\sin k(\lambda_n))$ with comultiplication Δ defined by

$$\Delta(Q_0^a) = Q_0^a \otimes 1 + 1 \otimes Q_0^a \tag{5.44}$$

$$\Delta(Q_1^a) = Q_1^a \otimes 1 + 1 \otimes Q_1^a + U \varepsilon^{abc} Q_0^b \otimes Q_0^c. \tag{5.45}$$

This representation is irreducible since $k(\lambda_i)$'s are real. Hence, we conclude that all the n-particle states of (5.36) can be constructed by applying the Yangian generators $Q_n^a(\nu)$ to the highest weight state

$$R_\uparrow(\lambda_1)^\dagger_\nu \ldots R_\uparrow(\lambda_n)^\dagger_\nu |0\rangle \tag{5.46}$$

which is clearly proportional to

$$c_\uparrow(k(\lambda_1, \nu))^\dagger \ldots c_\uparrow(k(\lambda_n, \nu))^\dagger |0\rangle \tag{5.47}$$

with $c_\alpha(k)^\dagger = \sum_j \mathrm{e}^{-\mathrm{i}jk} c^\dagger_{j\alpha}$. If we use $\hat{R}$ operators instead of R, we reach the similar conclusion with $k(\lambda)$ replaced by $p(\lambda)$ and the definition of the comultiplication changed to

$$\Delta'(Q_0^a) = Q_0^a \otimes 1 + 1 \otimes Q_0^a \tag{5.48}$$

$$\Delta'(Q_1^a) = Q_1^a \otimes 1 + 1 \otimes Q_1^a - U \varepsilon^{abc} Q_0^b \otimes Q_0^c. \tag{5.49}$$

5.3. Bound states

In order to investigate structures of bound states, let us begin with the two-particle bound states. Among two-particle states

$$C_{2a}(\lambda_1)_\nu C_{2b}(\lambda_2)_\nu |0\rangle \tag{5.50}$$

we should set $(a, b) = (2, 1), (1, 2)$ in order to obtain bound states, which follow from an explicit calculation of wavefunctions. In the former case, in which the eigenstate is calculated as

$$C_{22}(\lambda_1)_\nu C_{21}(\lambda_2)_\nu|0\rangle \propto \sum_{j,l} c^\dagger_{j\downarrow} c^\dagger_{l\uparrow}[\theta(j > l)(v(\lambda_1) - v(\lambda_2))\mathrm{e}^{-\mathrm{i}jk(\lambda_1,\nu)}\mathrm{e}^{-\mathrm{i}lk(\lambda_2,\nu)}$$
$$+\theta(j < l)(v(\lambda_1) - v(\lambda_2) + \mathrm{i}U)\mathrm{e}^{-\mathrm{i}jk(\lambda_1,\nu)}\mathrm{e}^{-\mathrm{i}lk(\lambda_2,\nu)}$$
$$+\theta(j < l)(-\mathrm{i}U)\mathrm{e}^{-\mathrm{i}lk(\lambda_1,\nu)}\mathrm{e}^{-\mathrm{i}jk(\lambda_2,\nu)}$$
$$+\delta_{jl}\mathrm{e}^{-\mathrm{i}j\{k(\lambda_1,\nu)+k(\lambda_2,\nu)\}}(v(\lambda_1) - v(\lambda_2) + \mathrm{i}U)F(\lambda_1, \lambda_2, \nu)]|0\rangle. \quad (5.51)$$

The condition for it to be a bound state is

$$v(\lambda_1) - v(\lambda_2) = -\mathrm{i}U \quad (5.52)$$
$$\mathrm{Im}\, k(\lambda_1, \nu) = -\mathrm{Im}\, k(\lambda_2, \nu) < 0. \quad (5.53)$$

Provided that these conditions hold, it follows that

$$C_{22}(\lambda_1)_\nu C_{21}(\lambda_2)_\nu = -C_{21}(\lambda_1)_\nu C_{22}(\lambda_2)_\nu \quad (5.54)$$

which implies that $(a, b) = (2, 1)$ and $(2, 1)$ cases give the same bound state. To summarize, among the two-particle states (5.50), there is only one bound state, which is achieved in the case $(a, b) = (2, 1)$ with the conditions (5.52), (5.53). We have not, however, succeeded to investigate this condition further due to the complicated form of the function $k(\lambda, \nu)$.

Due to this complicated form of $k(\lambda, \nu)$, we have not discovered general forms of multiparticle bound states or their creation operators. However, if we simply mimic the construction of bound state operators of the original Hubbard model in [21], we can formally make 'bound-state operators' by

$$C_2^{(2m)}(\lambda_1, \ldots, \lambda_{2m})_\nu = C_{22}(\lambda_1)_\nu C_{21}(\lambda_2)_\nu C_{22}(\lambda_3)_\nu C_{21}(\lambda_4)_\nu \ldots C_{22}(\lambda_{2m-1})_\nu C_{21}(\lambda_{2m})_\nu \quad (5.55)$$
$$D_{22}^{(2m)}(\lambda_1, \ldots, \lambda_{2m})_\nu = D_{22}(\lambda_1)_\nu D_{22}(\lambda_2)_\nu \ldots D_{22}(\lambda_{2m-1})_\nu D_{22}(\lambda_{2m})_\nu \quad (5.56)$$
$$R^{(2m)}(\lambda_1, \ldots, \lambda_{2m})^\dagger_\nu = C_2^{(2m)}(\lambda_1, \ldots, \lambda_{2m})_\nu D_{22}^{(2m)}(\lambda_1, \ldots, \lambda_{2m})^{-1}_\nu \quad (5.57)$$

where

$$k(\lambda_{2s}) + p(\lambda_{2s-1}) = \pi \pmod{2\pi} \quad (5.58)$$
$$\sin k(\lambda_{2s-1}) = \sin k(\lambda_1) + \frac{\mathrm{i}U(s-1)}{2} \qquad (s = 1, \ldots, m). \quad (5.59)$$

The commutation rules of $R^{(2m)\dagger}$ with $R^{(2n)\dagger}$, $R^\dagger$ or $Q^a_n(\nu)$ are the same as those in the case of the usual Hubbard model ((6.58)–(6.60) in [21]). However, the serious problem with this operator $R^{(2m)\dagger}$ is that we do not know for certain whether it produces physical states. As the case of two-particle bound states ($m = 1$) is already difficult to study, there is little hope that we can obtain a deep understanding of multiparticle bound-state operators.

6. Concluding remarks and discussion

In this paper we have introduced new integrable variant of the nearest-neighbour Hubbard model with variable range hopping. We have constructed it by the quantum inverse scattering method on the infinite interval at zero density, using the one-parameter deformation of the $\mathcal{L}$-matrix of the Hubbard model. By construction, together with the knowledge of the case of the Hubbard model studied earlier, this Hamiltonian is among an infinite number of commuting operators and thus integrable. Moreover, it commutes with operators $Q^a_n(\nu)$ $(n = 0, 1;\ a = x, y, z)$, which form a representation of the Y(su(2)) Yangian. If we take the Hamiltonian $\hat{H}'_{\rm long}$ or $\hat{H}''_{\rm long}$ instead of $\hat{H}_{\rm long}$, it is invariant under the partial particle–hole transformation and is $\mathrm{Y(su(2))} \oplus \mathrm{Y(su(2))}$ invariant. Normalized creation and annihilation operators of quasiparticles are explicitly constructed and are shown to form the Zamolodchikov–Faddeev algebra. Multiparticle scattering states are constructed with these operators, while bound states still require further study.

The merits of deriving the Hamiltonian along such lines are as follows. First, the existence of the Yangian symmetry can be established without any *ad hoc* methods. Second, the forms of multiparticle states upon the zero-density vacuum can be derived without any ansatz. They are calculated directly from actions of the elements of the monodromy matrix $\tilde{\mathcal{T}}(\lambda)_\nu$ on the vacuum. Derivation of multiparticle wavefunctions by making some kind of 'Bethe ansatz' is rather difficult due to the complicated structure of the Hamiltonian.

Although it is not easy to interpret the physical meaning of the term of the phase factor $\exp(\pm 2h(\nu)(1 - n_{i,-\sigma} - n_{j,-\sigma}))$ in the Hamiltonian, it would be interesting to investigate thermodynamic properties of this new integrable model and it will be studied in separate papers.

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Appendix. Expressions of the $\check{\mathcal{R}}$-matrix and the $\mathcal{L}$-matrix

In this appendix we present the expressions of the $\check{\mathcal{R}}$-matrix and the $\mathcal{L}$-matrix. The $\mathcal{L}$-matrix [6] is given as

$$\mathcal{L}_j(\lambda) = \mathrm{e}^{h(\lambda)\sigma^z \otimes \sigma^z/2} \left[\begin{pmatrix} -f_{j\uparrow}(\lambda) & \mathrm{i}c_{j\uparrow} \\ c^\dagger_{j\uparrow} & g_{j\uparrow}(\lambda) \end{pmatrix} \otimes_s \begin{pmatrix} f_{j\downarrow}(\lambda) & c_{j\downarrow} \\ \mathrm{i}c^\dagger_{j\downarrow} & -g_{j\downarrow}(\lambda) \end{pmatrix} \right] \mathrm{e}^{h(\lambda)\sigma^z \otimes \sigma^z/2} \tag{6.1}$$

with the grading $P(1) = 0$, $P(2) = 1$ and functions

$$f_{j\sigma}(\lambda) = (\mathrm{i}\cot\lambda)^{n_{j\sigma}} \sin\lambda \qquad g_{j\sigma}(\lambda) = (-\mathrm{i}\tan\lambda)^{n_{j\sigma}} \cos\lambda. \tag{6.2}$$

The $\check{\mathcal{R}}$-matrix is given as [6]

$$\check{\mathcal{R}}(\lambda,\mu) = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 & \mathrm{i}\rho_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}\rho_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho_3 & 0 & 0 & -\mathrm{i}\rho_6 & 0 & 0 & \mathrm{i}\rho_6 & 0 & 0 & -\rho_8 & 0 & 0 & 0 \\
0 & -\mathrm{i}\rho_{10} & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathrm{i}\rho_6 & 0 & 0 & \rho_5 & 0 & 0 & -\rho_7 & 0 & 0 & -\mathrm{i}\rho_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\rho_{10} & 0 & 0 \\
0 & 0 & -\mathrm{i}\rho_{10} & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\mathrm{i}\rho_6 & 0 & 0 & -\rho_7 & 0 & 0 & \rho_5 & 0 & 0 & \mathrm{i}\rho_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & -\mathrm{i}\rho_{10} & 0 \\
0 & 0 & 0 & -\rho_8 & 0 & 0 & \mathrm{i}\rho_6 & 0 & 0 & -\mathrm{i}\rho_6 & 0 & 0 & \rho_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}\rho_9 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}\rho_9 & 0 & 0 & \rho_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_1
\end{pmatrix} \tag{6.3}$$

where $\rho_j = \rho_j(\lambda, \mu)$ is defined by

$$
\begin{aligned}
\rho_1 &= (\mathrm{e}^{l}\cos\lambda\cos\mu + \mathrm{e}^{-l}\sin\lambda\sin\mu)\rho_2 \\
\rho_4 &= (\mathrm{e}^{l}\sin\lambda\sin\mu + \mathrm{e}^{-l}\cos\lambda\cos\mu)\rho_2 \\
\rho_9 &= (-\mathrm{e}^{l}\cos\lambda\sin\mu + \mathrm{e}^{-l}\sin\lambda\cos\mu)\rho_2 \\
\rho_{10} &= (\mathrm{e}^{l}\sin\lambda\cos\mu - \mathrm{e}^{-l}\cos\lambda\sin\mu)\rho_2 \\
\rho_3 &= \frac{\mathrm{e}^{l}\cos\lambda\cos\mu - \mathrm{e}^{-l}\sin\lambda\sin\mu}{\cos^2\lambda - \sin^2\mu}\rho_2 \\
\rho_5 &= \frac{-\mathrm{e}^{l}\sin\lambda\sin\mu + \mathrm{e}^{-l}\cos\lambda\cos\mu}{\cos^2\lambda - \sin^2\mu}\rho_2 \\
\rho_6 &= \frac{\mathrm{e}^{-2h(\mu)}\cos\lambda\sin\lambda - \mathrm{e}^{-2h(\lambda)}\cos\mu\sin\mu}{\cos^2\lambda - \sin^2\mu}\rho_2 \\
\rho_7 &= \rho_5 - \rho_4 \\
\rho_8 &= \rho_3 - \rho_1
\end{aligned}
$$

with $l = h(\lambda) - h(\mu)$. ρ_2 is an overall normalization factor of the $\check{\mathcal{R}}$-matrix and can be taken as an arbitrary function of λ and μ. The function $h(\lambda)$ is defined by (2.5). These matrices are identical with those in [6], except for the point that the spectral parameters λ and μ are shifted by $\pi/4$. These matrices satisfy the exchange relation (2.3). Its one-parameter deformed version is given by (2.14) with the $\check{\mathcal{R}}$-matrix presented above and the new $\mathcal{L}$-matrix is given by

$$
\begin{aligned}
\mathcal{L}_j(\lambda)_\nu^{11} &= \rho_1 n_{j\uparrow}n_{j\downarrow} - \mathrm{i}\rho_{10}(n_{j\uparrow} + n_{j\downarrow} - 2n_{j\uparrow}n_{j\downarrow}) - \rho_8(1-n_{j\uparrow})(1-n_{j\downarrow}) \\
\mathcal{L}_j(\lambda)_\nu^{12} &= -\mathrm{i}\rho_2 n_{j\uparrow}c_{j\downarrow} - \rho_6(1-n_{j\uparrow})c_{j\downarrow} \\
\mathcal{L}_j(\lambda)_\nu^{13} &= -\rho_2 c_{j\uparrow}n_{j\downarrow} + \mathrm{i}\rho_6 c_{j\uparrow}(1-n_{j\downarrow}) \\
\mathcal{L}_j(\lambda)_\nu^{14} &= \mathrm{i}\rho_3 c_{j\uparrow}c_{j\downarrow} \\
\mathcal{L}_j(\lambda)_\nu^{21} &= \rho_2 n_{j\uparrow}c_{j\downarrow}^{\dagger} - \mathrm{i}\rho_6(1-n_{j\uparrow})c_{j\downarrow}^{\dagger} \\
\mathcal{L}_j(\lambda)_\nu^{22} &= \rho_9 n_{j\uparrow}n_{j\downarrow} + \mathrm{i}\rho_4 n_{j\uparrow}(1-n_{j\downarrow}) - \mathrm{i}\rho_7(1-n_{j\uparrow})n_{j\downarrow} + \rho_9(1-n_{j\uparrow})(1-n_{j\downarrow}) \\
\mathcal{L}_j(\lambda)_\nu^{23} &= \rho_5 c_{j\uparrow}c_{j\downarrow}^{\dagger} \\
\mathcal{L}_j(\lambda)_\nu^{24} &= \rho_6 c_{j\uparrow}n_{j\downarrow} + \mathrm{i}\rho_2 c_{j\uparrow}(1-n_{j\downarrow}) \\
\mathcal{L}_j(\lambda)_\nu^{31} &= \mathrm{i}\rho_2 c_{j\uparrow}^{\dagger}n_{j\downarrow} + \rho_6 c_{j\uparrow}^{\dagger}(1-n_{j\downarrow}) \\
\mathcal{L}_j(\lambda)_\nu^{32} &= \rho_5 c_{j\uparrow}^{\dagger}c_{j\downarrow} \\
\mathcal{L}_j(\lambda)_\nu^{33} &= \rho_9 n_{j\uparrow}n_{j\downarrow} - \mathrm{i}\rho_7 n_{j\uparrow}(1-n_{j\downarrow}) + \mathrm{i}\rho_4(1-n_{j\uparrow})n_{j\downarrow} + \rho_9(1-n_{j\uparrow})(1-n_{j\downarrow}) \\
\mathcal{L}_j(\lambda)_\nu^{34} &= -\mathrm{i}\rho_6 n_{j\uparrow}c_{j\downarrow} + \rho_2(1-n_{j\uparrow})c_{j\downarrow} \\
\mathcal{L}_j(\lambda)_\nu^{41} &= -\mathrm{i}\rho_3 c_{j\uparrow}^{\dagger}c_{j\downarrow}^{\dagger} \\
\mathcal{L}_j(\lambda)_\nu^{42} &= -\mathrm{i}\rho_6 c_{j\uparrow}^{\dagger}n_{j\downarrow} + \rho_2 c_{j\uparrow}^{\dagger}(1-n_{j\downarrow}) \\
\mathcal{L}_j(\lambda)_\nu^{43} &= \rho_6 n_{j\uparrow}c_{j\downarrow}^{\dagger} + \mathrm{i}\rho_2(1-n_{j\uparrow})c_{j\downarrow}^{\dagger} \\
\mathcal{L}_j(\lambda)_\nu^{44} &= \rho_8 n_{j\uparrow}n_{j\downarrow} + \mathrm{i}\rho_{10}(n_{j\uparrow} + n_{j\downarrow} - 2n_{j\uparrow}n_{j\downarrow}) - \rho_1(1-n_{j\uparrow})(1-n_{j\downarrow})
\end{aligned}
$$

where $\rho_j = \rho_j(\lambda, \nu)$. One can easily check that $\mathcal{L}_j(\lambda)_{\nu=0} \propto \mathcal{L}_j(\lambda)$.

References

[1] Korepin V E and Eßler F H L (ed) 1994 *Exactly Solvable Models of Strongly Correlated Electrons* (Singapore: World Scientific)
[2] Lieb E H and Wu F Y 1968 *Phys. Rev. Lett.* **20** 1445
[3] Shastry B S 1986 *Phys. Rev. Lett.* **56** 1529
Shastry B S 1986 *Phys. Rev. Lett.* **56** 2453
[4] Shastry B S 1988 *J. Stat. Phys.* **50** 57
[5] Wadati M, Olmedilla E and Akutsu Y 1987 *J. Phys. Soc. Japan* **56** 1340
Olmedilla E and Wadati M 1988 *Phys. Rev. Lett.* **60** 1595
[6] Olmedilla E, Wadati M and Akutsu Y 1987 *J. Phys. Soc. Japan* **56** 2298
[7] Heilmann O J and Lieb E H 1971 *Ann. NY Acad. Sci.* **172** 583
Yang C N 1989 *Phys. Rev. Lett.* **63** 2144
Yang C N and Zhang S C 1990 *Mod. Phys. Lett.* B **4** 759
[8] Eßler F H L, Korepin V E and Schoutens K 1991 *Phys. Rev. Lett.* **67** 3848
Eßler F H L, Korepin V E and Schoutens K 1992 *Nucl. Phys.* B **372** 559
Eßler F H L, Korepin V E and Schoutens K 1992 *Nucl. Phys.* B **384** 431
[9] Eßler F H L and Korepin V E 1994 *Phys. Rev. Lett.* **72** 908
Eßler F H L and Korepin V E 1994 *Nucl. Phys.* B **426** 505
[10] Uglov D B and Korepin V E 1994 *Phys. Lett.* A **190** 238
[11] Ramos P B and Martins M J 1997 *J. Phys. A: Math. Gen.* **30** L195
[12] Yue R and Deguchi T 1997 *J. Phys. A: Math. Gen.* **30** 849
[13] Hikami K 1995 *Nucl. Phys.* B **441** 530
[14] Bouwknegt P and Schoutens K 1996 *Nucl. Phys.* B **482** 345
[15] Karowski M and Zapletal A 1994 *Nucl. Phys.* B **419** 567
Zapletal A and Karowski M 1994 *J. Phys. A: Math. Gen.* **27** 7419
[16] Gould M D, Hibberd K E, Links J R and Zhang Y Z 1996 *Phys. Lett.* A **212** 156
Bracken A J, Gould M D, Links J R and Zhang Y Z 1995 *Phys. Rev. Lett.* **74** 2768
Ge X Y, Gould M D, Zhang Y Z and Zhou H Q *Preprint* cond-mat/9711305
[17] Korepin V E, Bogoliubov N M and Izergin A G 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)
[18] Jimbo M and Miwa T 1995 *Algebraic Analysis of Solvable Lattice Models* (Providence, RI: American Mathematical Society)
[19] Shiroishi M and Wadati M 1995 *J. Phys. Soc. Japan* **64** 57
Yue R and Deguchi T 1997 *Preprint* cond-mat/9701207
[20] Murakami S and Göhmann F 1997 *Phys. Lett.* A **227** 216
[21] Murakami S and Göhmann F 1998 *Nucl. Phys.* B **512** 637
[22] Faddeev L D 1980 *Sov. Sci. Rev. Math. Phys.* C **1** 107
Thacker H B 1981 *Rev. Mod. Phys.* **53** 253
[23] Creamer D B, Thacker H B and Wilkinson D 1980 *Phys. Rev.* D **21** 1523
[24] Bernard D, Gaudin M, Haldane F D M and Pasquier V 1993 *J. Phys. A: Math. Gen.* **26** 5219
[25] Haldane F D M 1994 *Proc. 16th Taniguchi Symp. on Condensed Matter Physics* (New York: Springer)
[26] Murakami S and Wadati M 1996 *J. Phys. A: Math. Gen.* **29** 7903
[27] Murakami S and Wadati M 1996 *J. Phys. Soc. Japan* **65** 1227
[28] Drinfeld V G 1986 *Proc. 1986 Int. Congr. of Mathematics (Berkeley, CA)* (Providence, RI: American Mathematical Society)
[29] Chari V and Pressley A N 1990 *L'Enseignement Mathématique* **36** 267
Reine J 1991 *Ang. Math.* **417** 87
[30] Chari V and Pressley A N 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)
[31] Gebhard F and Ruckenstein A E 1992 *Phys. Rev. Lett.* **68** 244
Gebhard F, Girndt A and Ruckenstein A E 1994 *Phys. Rev.* B **49** 10 926
Bares P A and Gebhard F 1995 *J. Phys.: Condens. Matter* **7** 2285
Bares P A and Gebhard F 1995 *Europhys. Lett.* **29** 573
Göhmann F and Inozemtsev V 1996 *Phys. Lett.* A **214** 161
[32] Pu F C, Wu Y Z and Zhao B H 1987 *J. Phys. A: Math. Gen.* **20** 1173
[33] Göhmann F and Murakami S 1997 *J. Phys. A: Math. Gen.* **30** 5269
[34] Zamolodchikov A B and Zamolodchikov Al B 1979 *Ann. Phys.* **120** 253
[35] Smirnov F A 1992 *Form Factors in Completely Integrable Models of Quantum Field Theory* (Singapore: World Scientific)

[36] Khamitov I M 1985 *Theor. Math. Phys.* **62** 217
Khamitov I M 1985 *Theor. Math. Phys.* **63** 486
Khamitov I M 1988 *J. Sov. Math.* **40** 115

[37] Gómez C, Ruiz-Altaba M and Sierra G 1996 *Quantum Groups in Two-dimensional Physics* (Cambridge: Cambridge University Press)

[38] LeClair A and Smirnov F A 1992 *Int. J. Mod. Phys.* A **7** 2997